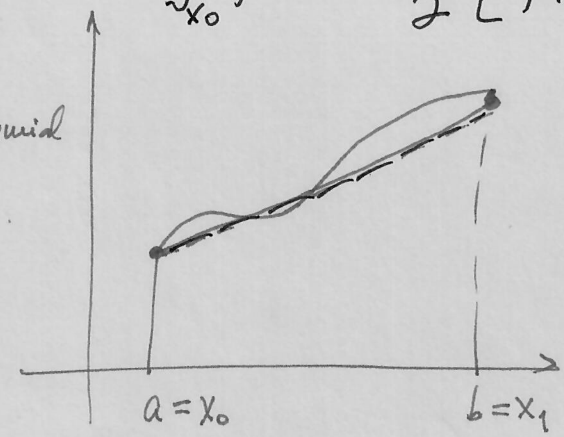
Two types of Quadrature rules.

Trapezoidal:

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}\left[f(x_0)+f(x_1)\right]$$

Interpolating polynomial degree one.



Degree of accuracy $n=1$

$h=b-a$

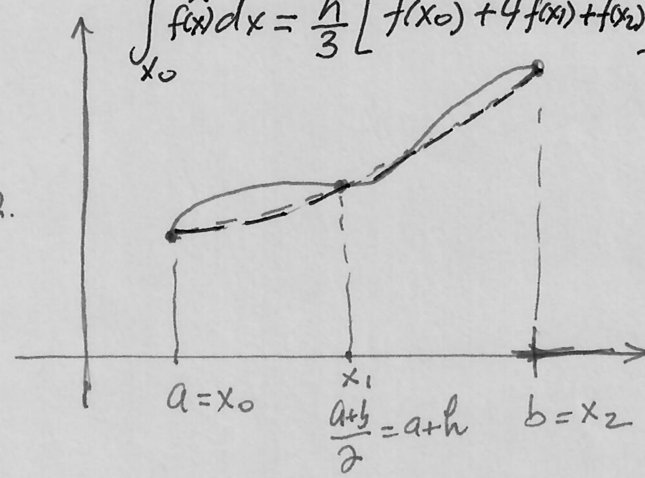
$$\text{Error} = -\frac{h^3}{12} f''(\xi)$$

Simpson:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

Degree of accuracy $n=2$

Interpolating polynomial degree 2.

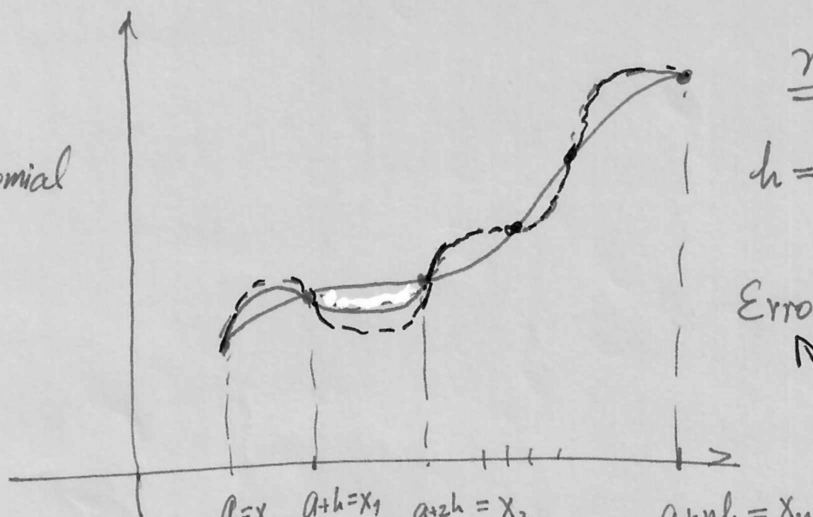


$$h=\frac{b-a}{2}$$

$$\text{Error} = -\frac{h^5}{90} f^{(4)}(\xi)$$

In general,

Interpolating polynomial degree n.



n

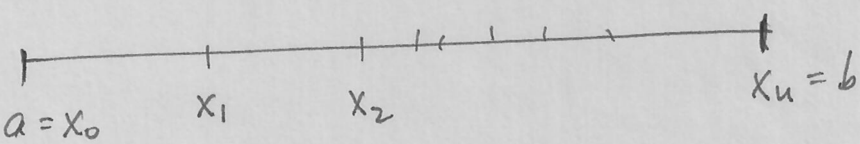
$$h=\frac{b-a}{n}$$

$$\text{Error} = \begin{cases} \text{option 1} & n \text{ even} \\ \text{option 2} & n \text{ odd.} \end{cases}$$

See theorem in following pages.

Above quadrature rules were obtained integrating Lagrange polynomials of degree 1 and degree 2 interpolating the equally spaced nodes.

In general, given $f(x) \in C^{n+1}[a,b]$ and a uniform partition



A quadrature formula for $f(x)$ can be obtained as

$$\int_a^b f(x)dx = \int_a^b P_n(x) + \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x-x_i)\, f^{(n+1)}(\xi(x))\,dx$$

$$= \int_a^b \sum_{i=0}^{n} f(x_i) L_i(x)\,dx +$$

or

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b L_i(x)\,dx + \text{Error} = \sum_{i=0}^{n} a_i f(x_i) + \text{Error}.$$

where $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$

Trapezoidal and Simpson are called Closed Newton-cotes formulas. The endpoints $x_0 = a$ and $x_n = b$ are included as nodes.

Theorem.– Consider $(n+1)$ points equally spaced on $[a,b]$ and a general closed Newton-Cotes formula defined over all these nodes including the endpoints.

Then,

i) If n is even and $f \in C^{n+2}[a,b]$, $n = 2,4,6\ldots$

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\ldots(t-n)\,dt$$

$h = \frac{b-a}{n}$ $\quad \xi \in (a,b)$

ii) If n is odd and $f \in C^{n+1}[a,b]$, $n = 1,3,5,\ldots$

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\ldots(t-n)\,dt.$$

$h = \frac{b-a}{n}$ $\quad \xi \in (a,b)$

Examples: Closed Newton-cotes formula.

$n=1$ Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] - \frac{h^3}{12} f''(\xi), \qquad x_0<\xi<x_1.$$

$$\boxed{h=b-a}$$

$n=2$ Simpson's rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi),$$

$$\boxed{h=\frac{b-a}{2}} \qquad \xi\in(x_0,x_2).$$

$n=3$ Simpson's three-eighths rule

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi)$$

$$\boxed{h=\frac{b-a}{3}} \qquad \xi\in(x_0,x_3)$$

Simpson quad. error

$$\int_0^2 t^2(t-1)(t-2)\,dt$$

$$= \int_0^2 t^2(t^2-3t+2)\,dt$$

$$= \int_0^2 (t^4-3t^3+2t^2)\,dt$$

$$\begin{array}{r} 15 \\ 12 \\ \hline 30 \\ 15 \\ \hline 180 \end{array}$$

$$= \left(\frac{t^5}{5} - \frac{3t^4}{4} + \frac{2t^3}{3}\right)\Bigg|_0^2 = \frac{32}{5} - \frac{\overset{12}{\cancel{48}}}{\cancel{4}} + \frac{16}{3}$$

$$= \frac{96-180+80}{15} = -\frac{4}{15}$$

then,

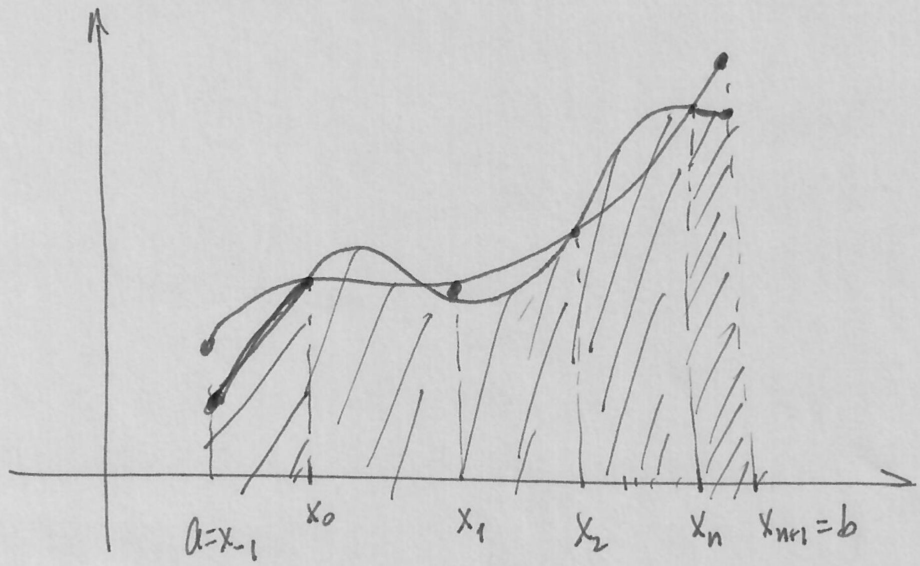
$$\text{Error} = \frac{h^5}{4!} f^{(4)}(\xi)\left(-\frac{4}{15}\right) = -\frac{h^5}{90} f^{(4)}(\xi).$$

This is obtained using the expression for the error in Thm 4.2 of Burden-Faires.

Second Type:

The endpoints $a = x_{-1}$, $b = x_{n+1}$ are not included.

They are called open Newton-Cotes



$(n+3)$ points in total.

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_{n+1}} f(x)\,dx \simeq \sum_{i=0}^{n} a_i f(x_i)$$

Theorem.- $(n+3)$ points equally spaced points

$x_{-1} = a$, $x_{n+1} = b$, $h = \dfrac{b-a}{n+2}$

Smaller step size than the one used in "closed" Newt-cotes.

then,

i) If n is even and $f \in C^{n+2}[a,b]$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2 (t-1)\cdots(t-n)\,dt$$

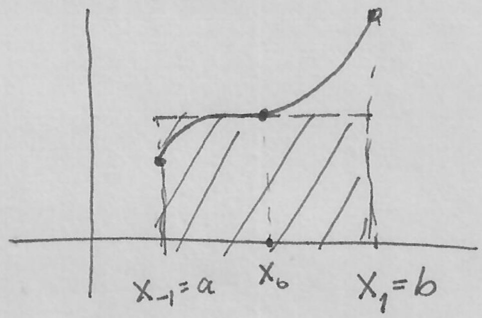
ii) If n is odd and $f \in C^{n+1}[a,b]$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t(t-1)\cdots(t-n)\,dt.$$

$\xi \in (a,b)$

Examples open-Newton-Cotes formulas

$n = 0$ Midpoint rule

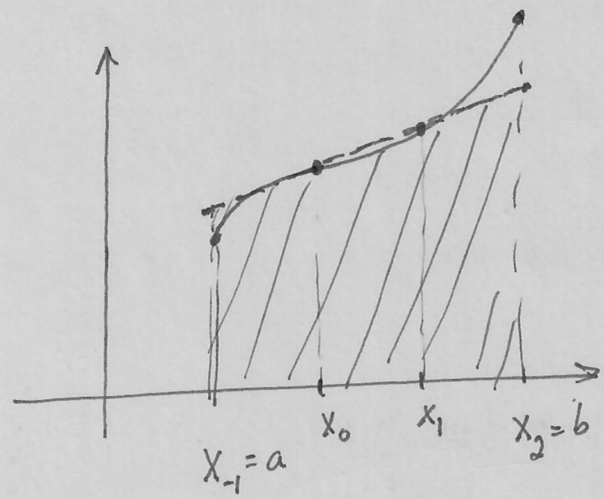


$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2h\, f(x_0) + \frac{h^3}{3} f''(\xi)$$

$$\boxed{h = \frac{b-a}{2}} \qquad \xi \in (x_{-1}, x_1)$$

$n = 1$

$$\boxed{h = \frac{b-a}{3}}$$

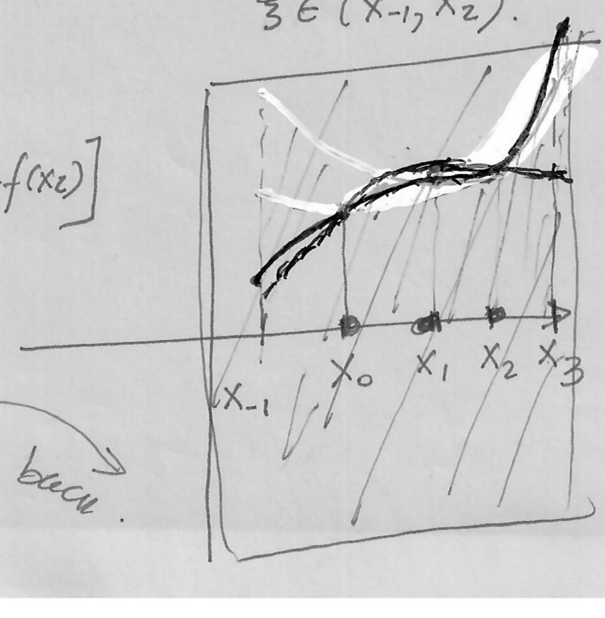


$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1) \right] + \frac{3h^3}{4} f''(\xi)$$

$$\xi \in (x_{-1}, x_2).$$

$n = 2$

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2) \right] + \frac{14h^5}{45} f^{(4)}(\xi).$$



back.

$$\boxed{h = \frac{b-a}{4}}$$